Homological mirror symmetry for open Riemann surfaces from pair-of-pants decompositions

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Punctured surfaces

Follow G. Mikhalkin:

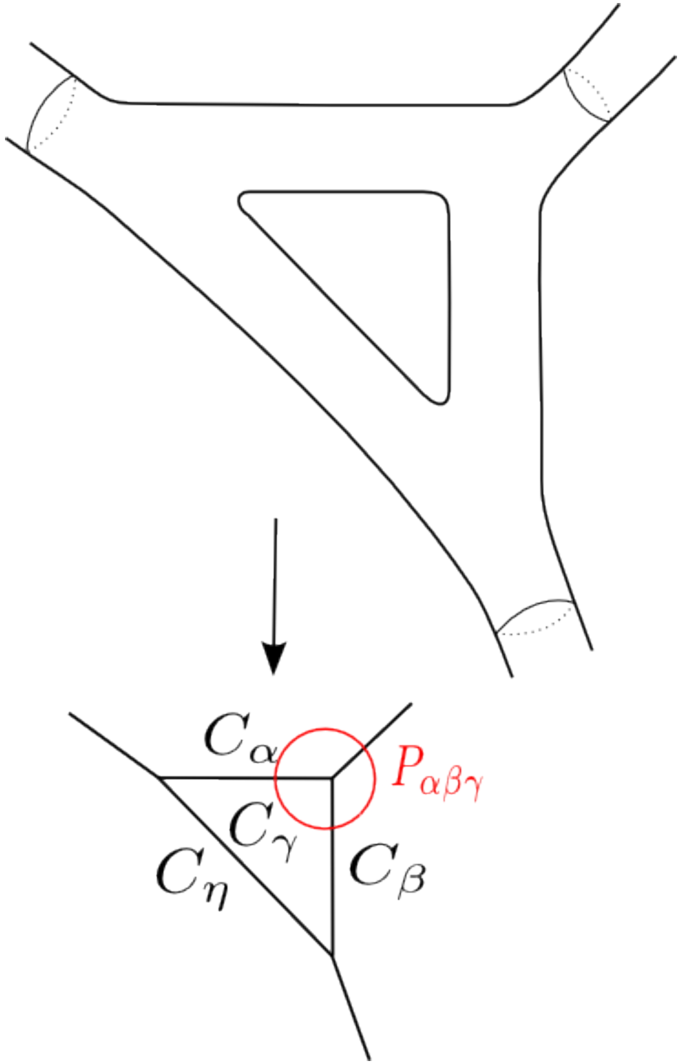
$C \in$ a degenerating family of hypersurfaces

$$C_t = \left\{ f_t(z) = \sum_{\alpha=(\alpha_1,\alpha_2)\in A\subset\mathbb{Z}^2} c_\alpha t^{-\nu(\alpha)} z_1^{\alpha_1} z_2^{\alpha_2} = 0 \right\} \subset (\mathbb{C}^*)^2$$

$\nu : \mathrm{Conv}(A) \to \mathbb{R}$ convex piecewise linear

$\mathrm{Log}_t(C_t) \xrightarrow{t\to\infty} \Gamma =$ singular locus of $L_\nu(\xi) = \max\{\langle \alpha, \xi\rangle - \nu(\alpha) | \alpha \in A\}$

$\mathbb{R}^2 \backslash \Gamma = \bigcup_{\alpha\in A} C_\alpha$



SYZ mirror symmetry for C [Abouzaid-Auroux-Katzarkov 1205.0053]

(Y, W)

toric $\dim_{\mathbb{C}} = 3$.

$\dim_{\mathbb{C}}(\mathrm{Crit}(W)) = 1$

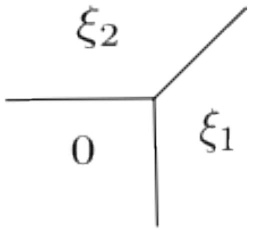
$$\Delta_Y = \{(\xi_1, \xi_2, \eta) \in \mathbb{R}^3 | \eta \geq L_\nu(\xi)\}$$

$$W = -z^{(0,0,1)} \in \mathcal{O}(Y)$$

$$W^{-1}(0) = D = \coprod_{\alpha \in A} D_\alpha$$

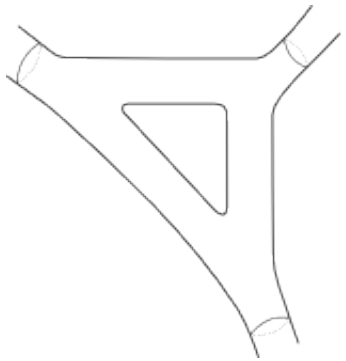
Example:

$C = \{1 + x_1 + x_2 = 0\}$

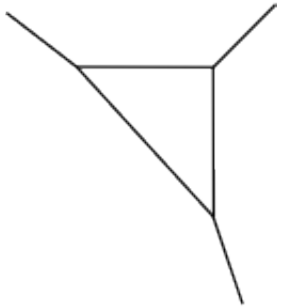


$(\mathbb{C}^3, xyz)$

Example:



$$C = \left\{ 1 + x_1 + x_2 + \frac{t}{x_1 x_2} = 0 \right\}$$



$$\begin{array}{ll} \mathcal{O}(-3) & s \\ \downarrow & \\ \mathbb{P}^2 & (x : y : z) \end{array} \quad , \; sxyz$$

Y is glued together from affine toric pieces according to the tropical hypersurface,

$$MF(Y, W) = \lim MF(\text{pieces})$$

Homological Mirror Symmetry

$$\mathcal{W}(C) \quad \cong \quad D^b_{sing}(D = W^{-1}(0))$$

[AAEKO 1103.4322
Bocklandt 1111.3392
…]

whenever C is noncompact

Matrix factorization $\qquad$ Category of singularities

$$MF(Y,W) \cong D^b_{sing}(D) = D^b(Coh(D))/Perf$$

$$T_\alpha(k) := \mathcal{O}(-D_\alpha)(k) \underset{t_{\alpha;0}}{\overset{t_{\alpha;1}}{\rightleftarrows}} \mathcal{O}(k) \quad \leftrightarrow \quad \mathrm{coker}(t_{\alpha;1}) = \mathcal{O}_{D_\alpha}(k)$$

Example: mirror of a pair of pants $= (\mathbb{C}^3, xyz)$, $\hat{R} = \mathbb{C}[x,y,z]$, $R = \hat{R}/\langle xyz \rangle$

$$\hat{R} \underset{yz}{\overset{x}{\rightleftarrows}} \hat{R} \quad \leftrightarrow \quad \{\cdots \xrightarrow{x} R \xrightarrow{yz} R \xrightarrow{x} R\} \to \hat{R}/\langle x \rangle$$

(2-periodic)

Objects of $\mathcal{W}(C)$

Abouzaid's generation

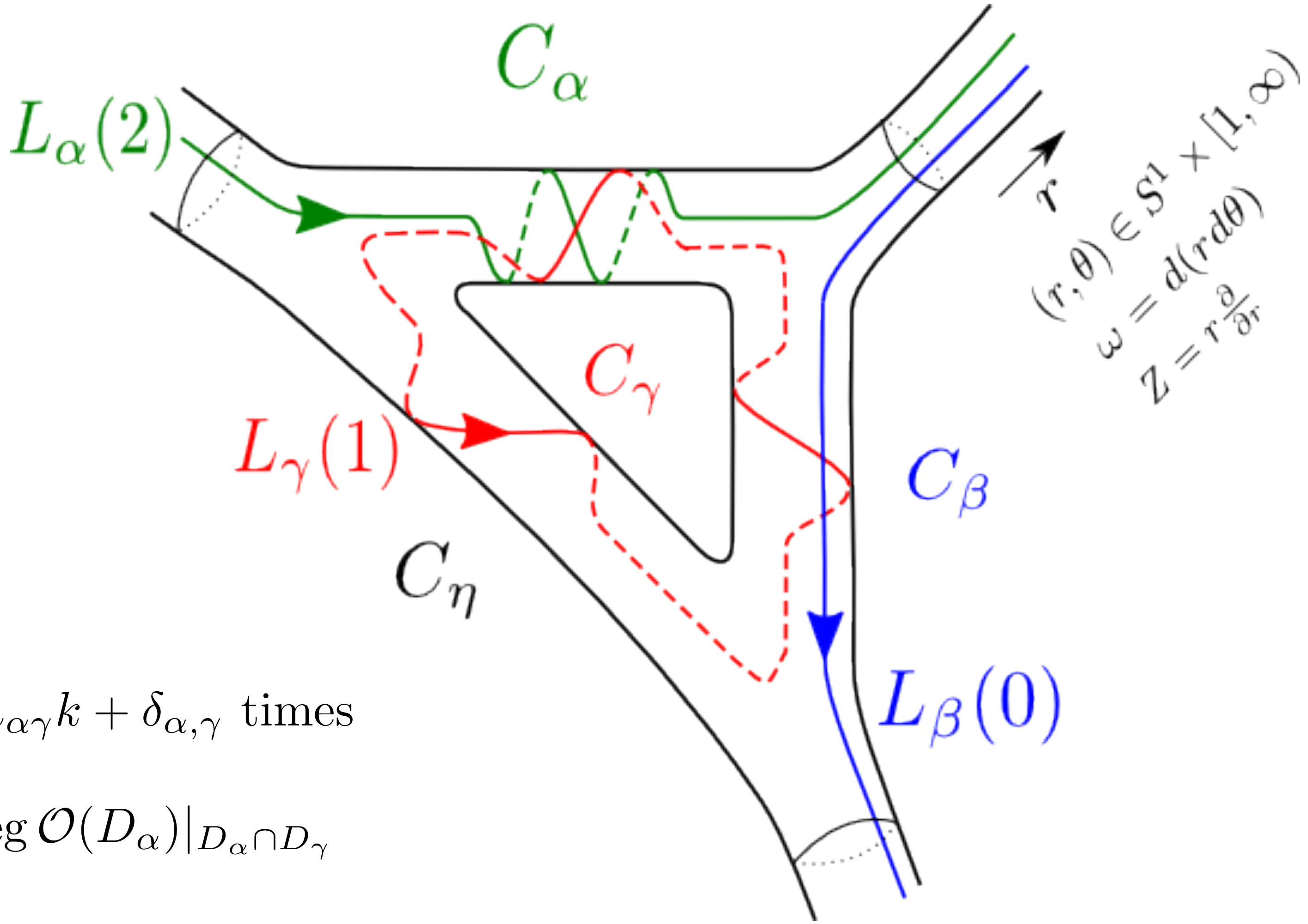
⇩

$\mathcal{W}(C)$ is split-generated by

$L_\alpha(k),\ \alpha \in A,\ k \in \mathbb{Z}.$

$L_\alpha(k)$ winds around each edge $k_{\alpha\gamma} = n_{\alpha\gamma}k + \delta_{\alpha,\gamma}$ times

$\delta_{\gamma,\alpha} - \delta_{\alpha,\gamma} = 1 + d_{\alpha,\gamma}$, where $d_{\alpha,\gamma} = \deg \mathcal{O}(D_\alpha)|_{D_\alpha \cap D_\gamma}$



e.g. $\delta_{\gamma,\alpha} = \delta_{\gamma,\beta} = 2$

$\delta_{\alpha,\gamma} = \delta_{\beta,\gamma} = 0$

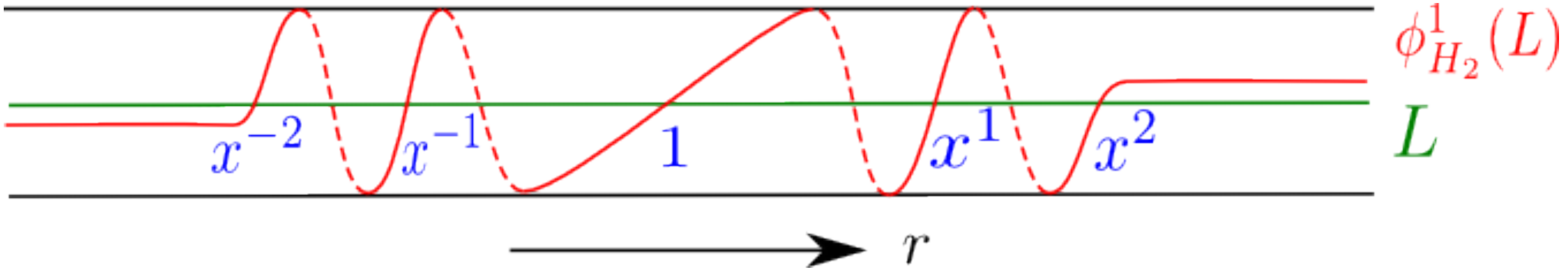
Wrapped Fukaya category [Abouzaid-Seidel 0712.3177]

Example: Cylinder $\mathbb{R} \times S^1$, $\omega = d(rd\theta)$

Model: choice of Hamiltonian $H = 2\pi|r|$ when $|r| \gg 1$, $H_n = nH$

Floer complexes: $CF^*(L_1, L_2; H_n) = \langle \phi^1_{H_n}(L_1) \cap L_2 \rangle$.

Figure below: generators of $CF^*(L, L; H_2)$



Wrapped Fukaya category

Morphism:

$$\hom(L_1, L_2) = CW^*(L_1, L_2)$$

$$= \varinjlim_n CF^*(L_1, L_2; H_n) = \coprod_{n=1}^{\infty} CF^*(L_1, L_2; nH)/\sim.$$

When all higher order continuation maps are trivial.

continuation maps: just inclusions in this example

In the example: $\hom(L, L) = \langle \bigcup_{i \in \mathbb{Z}} x^i \rangle$.

Wrapped Fukaya category

Product (and A_∞-products are similiar with more terms)

$$\mu^2_H : CF^*(L_1, L_2; nH) \otimes CF^*(L_0, L_1; nH) \to CF^*(L_0, L_2; 2nH)$$

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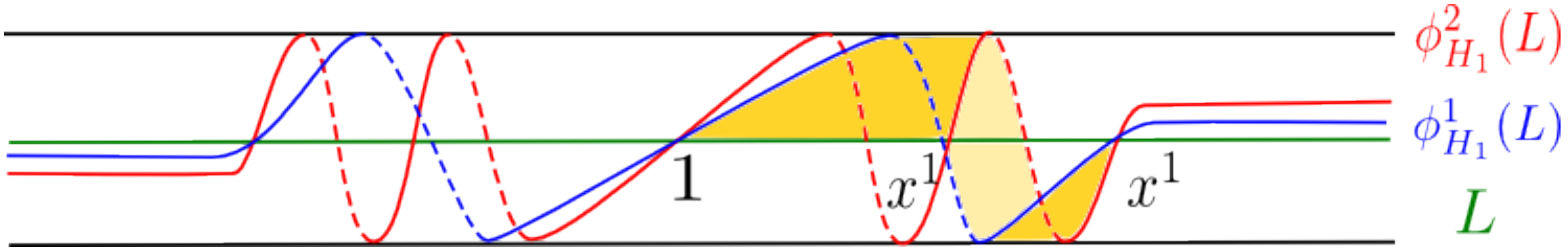
$$\mu^2 : CW^*(L_1, L_2) \otimes CW^*(L_0, L_1) \to CW^*(L_0, L_2)$$



$\mathcal{W}(\mathbb{R} \times S^1)$ generated by L

$Coh(\mathbb{C}^*)$ = finite modules over $\mathbb{C}[x, x^{-1}]$ generated by $\mathcal{O}$

$$CW^*(L, L) \cong \mathbb{C}[x, x^{-1}] \cong \text{End}(\mathcal{O})$$

Can we compute $\mathcal{W}(C)$ from $\mathcal{W}(P_{\alpha\beta\gamma})$'s ?

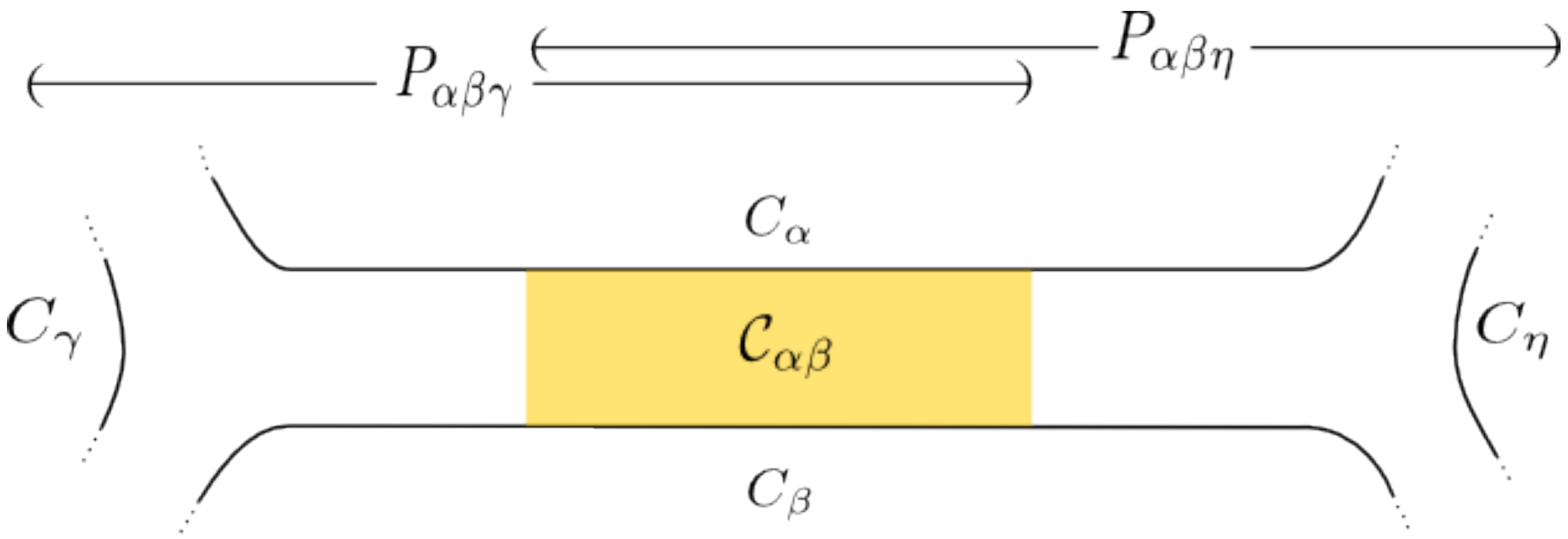
Main theorem [L.]:

- $\mathcal{W}(C)$ is split-generated by the objects $L_\alpha(k), \alpha \in A, k \in \mathbb{Z}$.
- In a suitable model for $\mathcal{W}(C)$, the morphism complex between any two objects, $L_\alpha(k)$ and $L_\beta(l)$, is generated by

$$\mathcal{X}\left(L_\alpha(k), L_\beta(l)\right) = \left(\bigcup \mathcal{X}_{P_{\alpha\beta\gamma}}\left(L_\alpha(k), L_\beta(l)\right)\right) / \sim$$

with $x \in \mathcal{X}_{P_{\alpha\beta\gamma} \cap \mathcal{C}_{\alpha\beta}} \sim y \in \mathcal{X}_{P_{\alpha\beta\eta} \cap \mathcal{C}_{\alpha\beta}}$ whenever $\rho^{\gamma}_{\alpha\beta}(x) = \rho^{\eta}_{\alpha\beta}(y)$.

- In this model, the A_∞-products in $\mathcal{W}(C)$ are given by those in the pairs of pants.



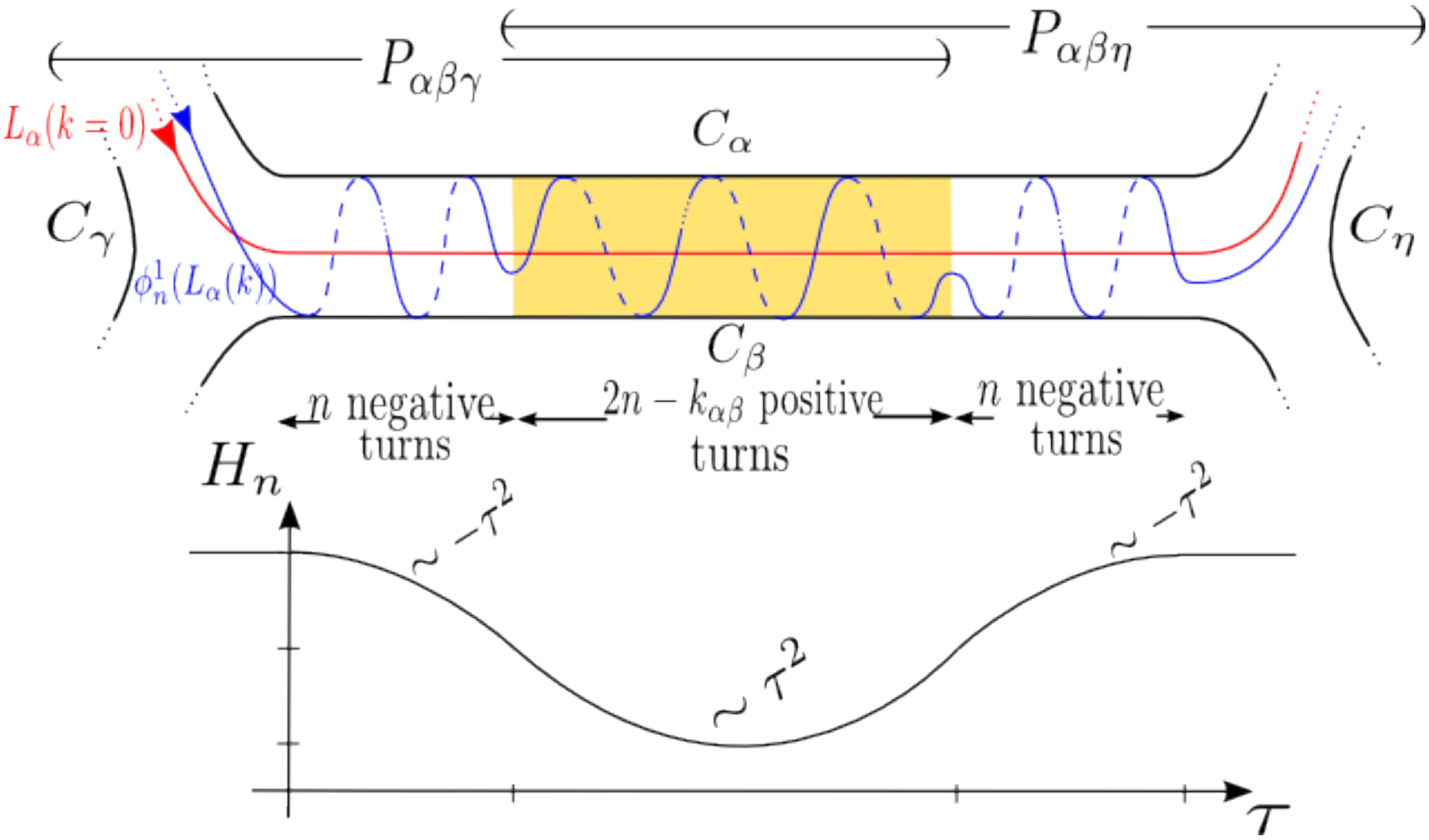
Restriction maps

$$\bigoplus_{L_i, L_j} CW^*_{P_{\alpha\beta\gamma}}(L_i, L_j)$$

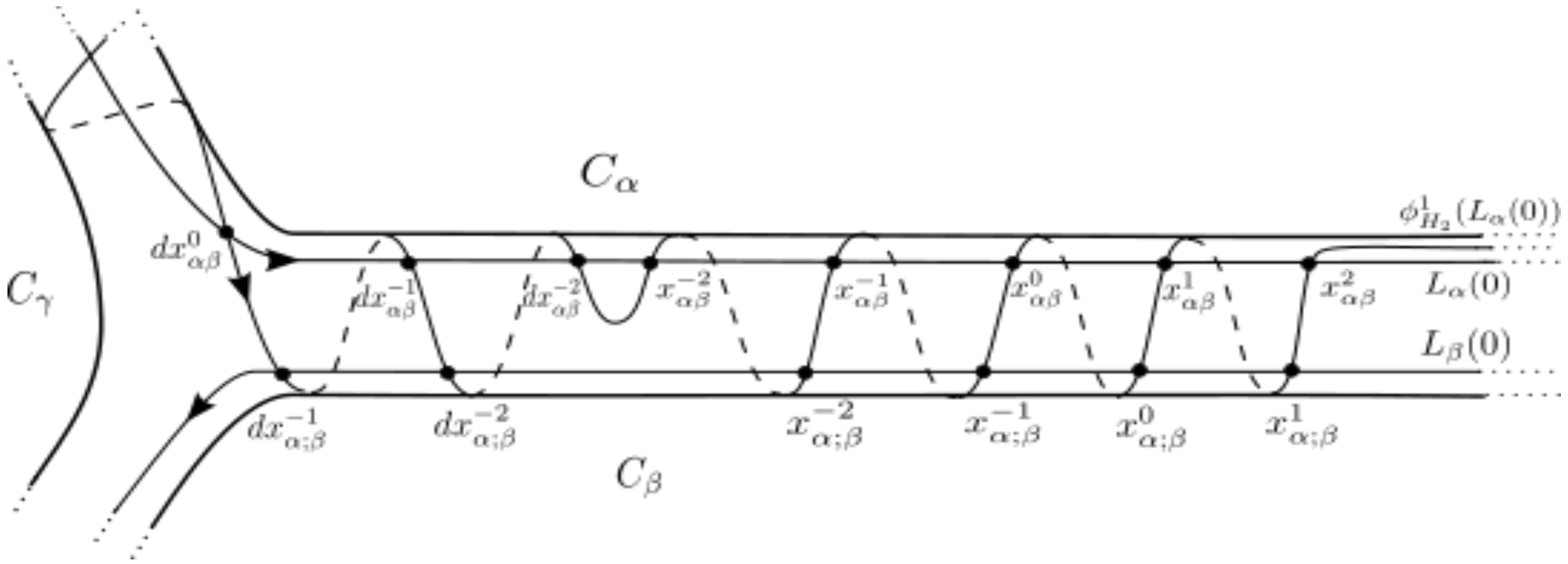
$$\downarrow \rho^{\gamma}_{\alpha\beta}$$

$$\bigoplus_{L_i, L_j} CW^*_{P_{\alpha\beta\eta}}(L_i, L_j) \xrightarrow{\rho^{\eta}_{\alpha\beta}} \bigoplus_{L_i, L_j} CW^*_{\mathcal{C}_{\alpha\beta}}(L_i, L_j)$$

Ingredient: Hamiltonian perturbation



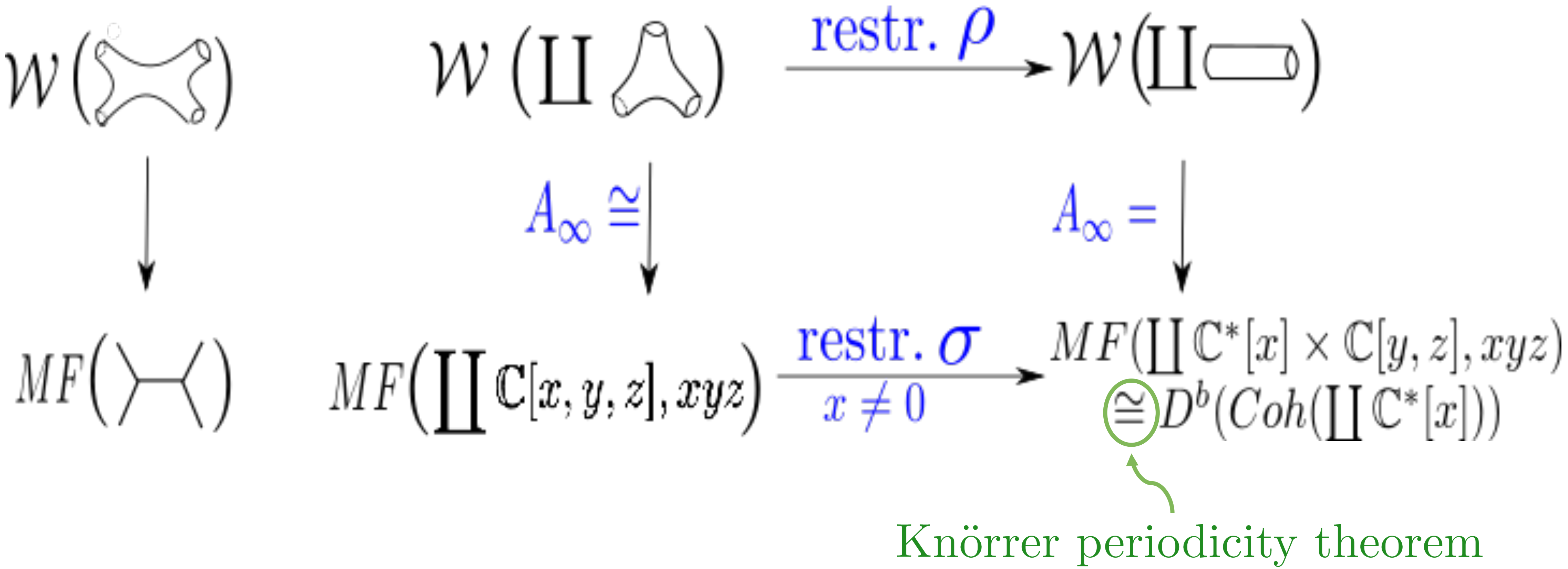
Equivalence relations

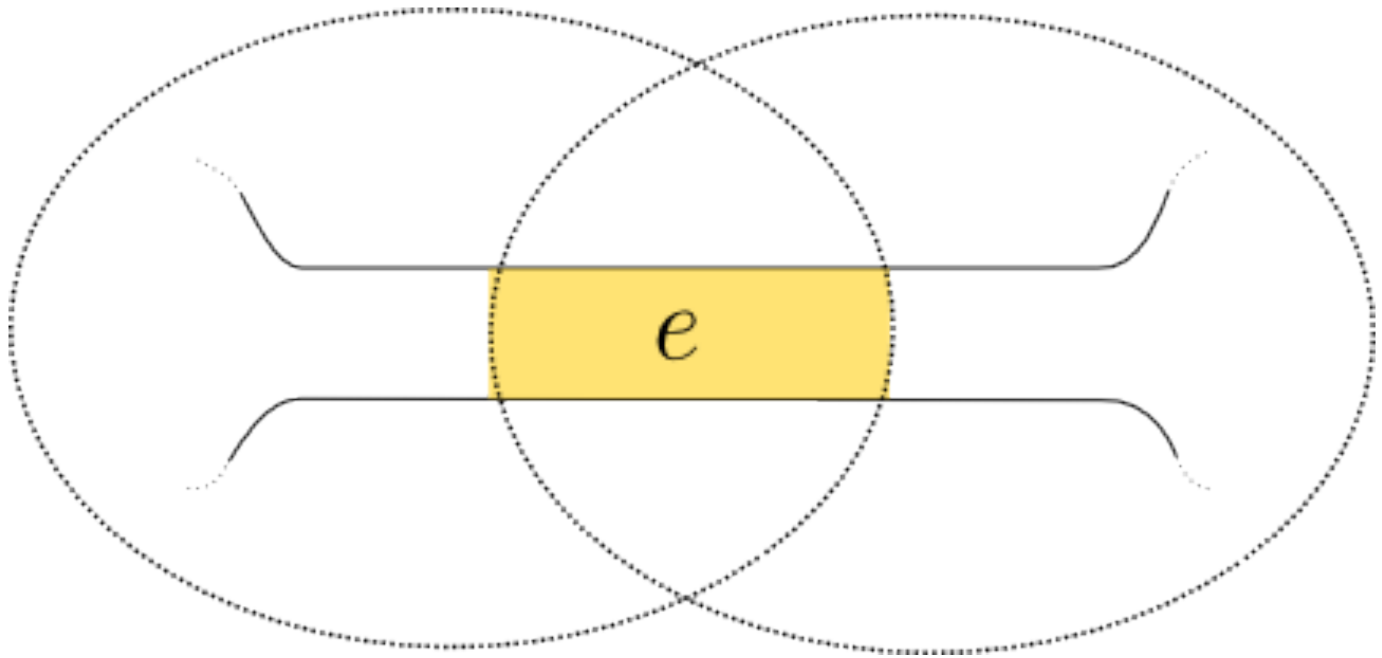


$$\rho^\gamma_{\alpha\beta}\left(x^i_{\alpha\beta}\right) = \rho^\eta_{\alpha\beta}\left(\tilde{x}^{n_{\alpha\beta}(l-k)-i}_{\alpha\beta}\right) \Rightarrow x^i_{\alpha\beta} \sim \tilde{x}^{n_{\alpha\beta}(l-k)-i}_{\alpha\beta} \in \mathcal{X}(L_\alpha(k), L_\alpha(l))$$

$$\rho^\gamma_{\alpha\beta}\left(x^i_{\alpha;\beta}\right) = \rho^\eta_{\alpha\beta}\left(\tilde{x}^{n_{\alpha\beta}(l-k)+d_{\alpha,\beta}-i}_{\alpha;\beta}\right) \Rightarrow x^i_{\alpha;\beta} \sim \tilde{x}^{n_{\alpha\beta}(l-k)+d_{\alpha,\beta}-i}_{\alpha;\beta} \in \mathcal{X}(L_\alpha(k), L_\beta(l))$$

Homological Mirror Symmetry





Pick a split-generating set of Lagrangians so that for each L:

- $L \cap e = \bigcup(\text{disjoint arcs})$, each arc intersects each circle fiber of e just once.
- For any two arcs, the portion of L in the complement of e connected by these two arcs cannot be homotopically trivial.